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On the variational solution of electromagnetic problems in lossy anisotropic inhomogeneous media

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Abstract. Some of the basic developments in the variational treatment of electromagnetic field problems in anisotropic inhomogeneous media are reviewed. The field is assumed to be time harmonic and satisfies appropriate Maxwell's equations. A vector variational principle in terms of electric and magnetic field vectors for lossy media is developed. The natural interface and boundary conditions associated with the principle are investigated and further modifications needed to implement more required conditions are presented. The results extend the class of problems for which the variational approach is useful in addition to facilitating the computations involved.

1. Introduction

Exact solutions of the problems of physics may be obtained for only a limited class of problems. For most cases, it is often necessary to solve partial differential or integral equations subject to complicated boundary conditions. By their nature, these equations seldom admit simple rigorous solution and we are led more and more to the use of approximate methods. Indeed, even for the problems which can be solved exactly, it may be more convenient to employ approximate methods, since the evaluation of the exact solution may be much too complicated. Among many approximate methods which can be used, variational methods occupy a prominent place (Mikhlin 1964, Kantorovic and Krylov 1958).

Many problems in physics can be characterised by variational principles (Morse and Feshbach 1953, chap 3, and Landau and Lifshitz 1971, chap 2). These problems are related to the minimisation of a variational integral, which often represents an energy of the system. The variational principle may succinctly summarise the equations, and allows insight into the effect of different parameters involved besides providing a useful means for approximating the solution.

The vector variational formulation of Maxwell's equations provides a useful method for solving a wide class of problems which would be intractable if a classical differential equation approach was used and which could not be formulated in terms of scalar variational principles. Berk (1956) presented variational expressions for the resonance frequency of a cavity and the propagation constant in a guiding structure. He considered inhomogeneous anisotropic media whose permittivity and permeability

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tensors are Hermitian. He pointed out that the reaction concept (Rumsey 1954), which yields variational expressions, would be appropriate for lossy media provided the media tensors are symmetric. Berk's results for the Hermitian media were re-derived by Cairo and Kahan (1965) using appropriate adjoint operators and inner products. English (1971) used Berk's formulation in terms of both electric and magnetic fields to study propagation through inhomogeneously loaded cylindrical waveguides. English and Young (1971) used the electric field formulation in a similar study and presented a discussion on the advantages and disadvantages of using one field vector rather than both electric and magnetic vectors. Konrad (1976) presented a review on the subject and obtained results corresponding to those of Berk for the electric, the magnetic as well as the magnetic potential vectors. He also discussed the problem of uniqueness of the solution. Morishita and Kumagai (1977) have recently re-derived Berk's result using the principle of least action. Alternative expressions for the propagation constant in a loss-free inhomogeneous waveguide with wall impedance were derived by Kurokawa (1962). These expressions were later used by Matsuhara and Kumagai (1974) and by Ohtaka *et al* (1976). The disadvantage of these expressions is that they include second derivatives rather than first derivatives of trial fields.

Most of the above developments are restricted to loss-free media. The natural boundary conditions resulting from the formulations, and means to implement some required—but not natural—conditions, still need further study.

This paper presents variational formulation for the field in a lossy anisotropic inhomogeneous medium and in the presence of sources. An adjoint operator is introduced which may be used for general lossy media and which reduces to the more convenient complex conjugate operator for Hermitian case. The boundary and interface conditions are investigated. Possible modifications to include some of these conditions in the formulation are presented. For the field in cylindrical guiding structures, a variational principle for the propagation constant is derived directly from the three-dimensional treatment of the fields.

2. Field equations and the variational expression

The linear medium under consideration has permittivity $\tilde{\epsilon}$ and permeability $\tilde{\mu}$ which are tensor functions of position. The medium occupies a volume V which is bounded by a surface S possibly extending to infinity. The electromagnetic field within the medium for harmonic time variation satisfies Maxwell's equations

$$\nabla \wedge \boldsymbol{E} = -\mathrm{i}\omega \boldsymbol{B}, \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

$$\nabla \wedge \boldsymbol{H} = \mathrm{i}\omega \boldsymbol{D} + \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{D} = \rho \tag{2}$$

and the constitutive relations

$$\boldsymbol{D} = \tilde{\epsilon} \cdot \boldsymbol{E}, \qquad \boldsymbol{B} = \tilde{\mu} \cdot \boldsymbol{H} \tag{3}$$

where ω is the angular frequency, $\boldsymbol{J}$ is the current source density assumed to exist away from boundaries or interfaces, $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic field vectors, $\boldsymbol{D}$ and $\boldsymbol{B}$ the electric displacement and the magnetic induction, respectively.

In order to consider general lossy media, we introduce an adjoint operator which is the transpose operator, in which $\tilde{\epsilon}$ and $\tilde{\mu}$ are replaced by their transposes, and the time dependence by its complex conjugate. Let the solution of the adjoint problem under conditions similar to the original problem be $(\boldsymbol{E}^a, \boldsymbol{H}^a)$ with current sources equal to (+) or (−) the complex conjugates of the original sources. With this choice, the solution of the adjoint problem when media tensors are Hermitian reduces to (+) or (−) the complex conjugate of the original problem.

We define a volume inner product and surface product by

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iiint_V \boldsymbol{A} \cdot \boldsymbol{B}^a \, \mathrm{d}V \tag{4}$$

and

$$(\boldsymbol{A}, \boldsymbol{B})_S = \iint_S \boldsymbol{A} \cdot (\hat{\boldsymbol{n}} \wedge \boldsymbol{B}^a) \, \mathrm{d}S \tag{5}$$

where $\hat{\boldsymbol{n}}$ is the outward unit vector to S. A variational expression may then be written in the form

$$F = \langle \nabla \wedge \boldsymbol{E}, \boldsymbol{H} \rangle + \langle \nabla \wedge \boldsymbol{H}, \boldsymbol{E} \rangle + \mathrm{i}\omega (\langle \boldsymbol{B}, \boldsymbol{H} \rangle - \langle \boldsymbol{D}, \boldsymbol{E} \rangle) - \langle \boldsymbol{J}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{J} \rangle. \tag{6}$$

Upon taking the first variation of F with respect to $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}^a, \boldsymbol{H}^a)$ and applying Green's identity

$$\iiint_V \boldsymbol{A} \cdot \nabla \wedge \boldsymbol{B} \, \mathrm{d}V \equiv \iiint_V \boldsymbol{B} \cdot \nabla \wedge \boldsymbol{A} \, \mathrm{d}V - \iint_S \boldsymbol{B} \cdot (\hat{\boldsymbol{n}} \wedge \boldsymbol{A}) \, \mathrm{d}S \tag{7}$$

we get

$$\delta F = \langle \nabla \wedge \boldsymbol{E} + \mathrm{i}\omega \boldsymbol{B}, \delta \boldsymbol{H} \rangle + \langle \nabla \wedge \boldsymbol{H} - \mathrm{i}\omega \boldsymbol{D} - \boldsymbol{J}, \delta \boldsymbol{E} \rangle + \langle \delta \boldsymbol{H}, \nabla \wedge \boldsymbol{E} + \mathrm{i}\omega \boldsymbol{B} \rangle$$
$$+ \langle \delta \boldsymbol{E}, \nabla \wedge \boldsymbol{H} - \mathrm{i}\omega \boldsymbol{D} - \boldsymbol{J} \rangle - [(\delta \boldsymbol{E}, \boldsymbol{H})]_{S_i} - [(\delta \boldsymbol{H}, \boldsymbol{E})]_{S_i} \tag{8}$$

where $[\kappa]_{S_i}$ represents the variation of κ across S_i. These quantities result from the surface integration along both sides of any surface of discontinuity S_i. The first variation of F vanishes if Maxwell's equations are satisfied in addition to the natural conditions that at any surface of discontinuity

$$[\hat{\boldsymbol{n}} \wedge \boldsymbol{H}] = 0 \tag{9}$$

and

$$[\hat{\boldsymbol{n}} \wedge \boldsymbol{E}] = 0. \tag{10}$$

This implies that both tangential electric and magnetic fields are continuous.

If the boundary is a perfect conductor, (10) implies that the required condition is natural but (9) would yield an unphysical boundary condition. To remedy this situation, as suggested by Berk's work, we may add the surface product $(\boldsymbol{E}, \boldsymbol{H})_S$ to equation (6).

The concept of perfect magnetic conductors is very useful when surfaces of symmetry are encountered. On such conductors the tangential magnetic field must vanish. If the boundary condition is of a mixed type, i.e. implying a perfect electric conductor on a part S_{I} of S and a perfect magnetic conductor on the remainder S_{II}, one may add $(\boldsymbol{E}, \boldsymbol{H})_{S_{\mathrm{I}}} + (\boldsymbol{H}, \boldsymbol{E})_{S_{\mathrm{II}}}$ to (6) to yield the required condition. Thus we have

$$F = \langle \nabla \wedge \boldsymbol{E}, \boldsymbol{H} \rangle + \langle \nabla \wedge \boldsymbol{H}, \boldsymbol{E} \rangle + \mathrm{i}\omega (\langle \boldsymbol{B}, \boldsymbol{H} \rangle - \langle \boldsymbol{D}, \boldsymbol{E} \rangle) \langle \boldsymbol{J}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{J} \rangle + (\boldsymbol{E}, \boldsymbol{H})_{S_{\mathrm{I}}} + (\boldsymbol{H}, \boldsymbol{E})_{S_{\mathrm{II}}} \tag{11}$$

with (9) and (10) as the associated natural boundary conditions at surfaces of discontinuities and

$$\hat{\boldsymbol{n}} \wedge \boldsymbol{E} = 0 \qquad \text{on } S_{\mathrm{I}} \tag{12}$$

and

$$\hat{\boldsymbol{n}} \wedge \boldsymbol{H} = 0 \qquad \text{on } S_{\mathrm{II}}. \tag{13}$$

3. Variational expression for the angular frequency

Having established that (6) is the proper variational expression for the fields, the appropriate stationary expression for the angular frequency ω may be easily obtained. Taking the first variation of (6) with field vectors as well as ω as variational parameters, we get

$$\delta F = \text{RHS}(8) + \mathrm{i}\delta\omega(\langle \boldsymbol{B}, \boldsymbol{H}\rangle - \langle \boldsymbol{D}, \boldsymbol{E}\rangle) \tag{14}$$

where RHS(8) is the right-hand side of equation (8). From (14) it follows that upon taking $F = 0$, (6) represents a variational principle for ω with (9) and (10) as natural boundary conditions and we may write

$$\mathrm{i}\omega = \frac{\langle \nabla \wedge \boldsymbol{E}, \boldsymbol{H}\rangle + \langle \nabla \wedge \boldsymbol{H}, \boldsymbol{E}\rangle - \langle \boldsymbol{J}, \boldsymbol{E}\rangle - \langle \boldsymbol{E}, \boldsymbol{J}\rangle}{\langle \boldsymbol{D}, \boldsymbol{E}\rangle - \langle \boldsymbol{B}, \boldsymbol{H}\rangle}. \tag{15}$$

Depending on the required boundary condition, a proper integral may be added to the numerator of (15) in order to satisfy such a condition as previously demonstrated in the second part of the paper. For the source-free case, (15) agrees with Harrington's expression (Harrington 1968) and reduces in the loss-less case to Berk's result (Berk 1956).

4. Variational expression for the propagation constant

Let the electromagnetic fields under consideration exist in a guiding structure of cross sectional area S and contour C. The permeability and permittivity tensors are assumed to be functions of the transverse direction (t) only. The dependence on the longitudinal direction z may be taken as $\exp(-\mathrm{i}\nu z)$, where ν is the propagation constant. The adjoint field is assumed to vary as $\exp(\mathrm{i}\nu z)$ which is equivalent to introducing a special adjoint ∇ operator as used by Cairo and Kahan (1965). Instead of developing a variational principle for ν from the start, a direct derivation is presented depending on the analysis of previous sections.

We utilise the vector identity (van Bladel 1964)

$$\iint_S \boldsymbol{A} \cdot \nabla_{\mathrm{t}} \wedge \boldsymbol{B}\, \mathrm{d}S = \iint_S \boldsymbol{B} \cdot \nabla_{\mathrm{t}} \wedge \boldsymbol{A}\, \mathrm{d}S - \int_C \boldsymbol{B} \cdot (\hat{\boldsymbol{n}} \wedge \boldsymbol{A})\, \mathrm{d}l \tag{16}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are three-dimensional vectors, $\hat{\boldsymbol{n}}$ is a unit outward vector normal to C and ∇_{t} represents the Laplacian operator in the transverse direction. With

$$\boldsymbol{A} = \boldsymbol{A}(\mathrm{t}) \exp(-\mathrm{i}\nu z) \qquad \text{and} \quad \boldsymbol{B} = \boldsymbol{B}(\mathrm{t}) \exp(\mathrm{i}\nu z)$$

and using $\nabla = (\partial/\partial z)\hat{z} + \nabla_t$ where $\hat{z}$ is a unit vector along z, it follows that

$$\iint_S \boldsymbol{A} \cdot \nabla \wedge \boldsymbol{B} \, \mathrm{d}S = \iint_S \boldsymbol{B} \cdot \nabla \wedge \boldsymbol{A} \, \mathrm{d}S - \int_C \boldsymbol{B} \cdot (\hat{\boldsymbol{n}} \wedge \boldsymbol{A}) \, \mathrm{d}l \tag{17}$$

which is identical to the three-dimensional identity (7) with surface integrals replacing volume integrals and line integrals taking the place of the surface integrals. As a consequence, results corresponding to those previously obtained in § 2 follow directly. For example equation (11) is now replaced by

$$\begin{aligned} F = &\langle \nabla \wedge \boldsymbol{E}, \boldsymbol{H} \rangle + \langle \nabla \wedge \boldsymbol{H}, \boldsymbol{E} \rangle + \mathrm{i}\omega(\langle \boldsymbol{B}, \boldsymbol{H} \rangle - \langle \boldsymbol{D}, \boldsymbol{E} \rangle) \\ &- \langle \boldsymbol{J}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{J} \rangle + (\boldsymbol{E}, \boldsymbol{H})_{C_{\mathrm{I}}} + (\boldsymbol{H}, \boldsymbol{E})_{C_{\mathrm{II}}} \end{aligned} \tag{18}$$

where the inner products are now defined by

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_S \boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{a}} \, \mathrm{d}S \tag{19}$$

$$(\boldsymbol{A}, \boldsymbol{B})_C = \int_C \boldsymbol{A} \cdot (\hat{\boldsymbol{n}} \wedge \boldsymbol{B}^{\mathrm{a}}) \, \mathrm{d}S. \tag{20}$$

The natural boundary conditions which follow from (18) are the continuity of both tangential electric and magnetic fields at interface boundaries and

$$\hat{\boldsymbol{n}} \wedge \boldsymbol{E} = 0 \qquad \text{on } C_{\mathrm{I}} \tag{21}$$

$$\hat{\boldsymbol{n}} \wedge \boldsymbol{H} = 0 \qquad \text{on } C_{\mathrm{II}}. \tag{22}$$

As previously demonstrated in § 3, a variational expression for a parameter ν (or ω) follows from (18) upon putting $F = 0$.

5. Conclusions

The paper presents variational principles for the electromagnetic fields, angular frequency and the propagation constant in a guiding structure. The media considered are linear lossy anisotropic and inhomogeneous. A proper adjoint operator and convenient inner products are introduced. The natural boundary conditions resulting from the formulation are discussed and means to satisfy additional required conditions are presented. In particular, mixed type boundary conditions are made natural. Inclusion of the required conditions as natural greatly extends the range of the applicability of the variational approach and permits the use of simpler expansion functions. This consequently may result in simpler integrals to be evaluated. It is shown that the continuity of tangential electric as well as magnetic fields at interfaces is guaranteed as a natural boundary condition.

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